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3 (Sem-3/CBCS) MAT HC 1

2023

**MATHEMATICS**

(Honours Core)

Paper : MAT-HC-3016

**(Theory of Real Functions)**

Full Marks : 80

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

1. Answer the following questions :  $1 \times 10 = 10$

(a) Is 0 a cluster point of  $(0,1)$  ?

(b) "If the limit of a function  $f$  at a point  $C$  of its domain does not exist, then  $f$  diverges at  $C$ ." (Write True or False)

(c) Define  $\lim_{x \rightarrow c} f(x) = \infty$ , where  $A \subseteq \mathbb{R}$  and  $f : A \rightarrow \mathbb{R}$  and  $C \in \mathbb{R}$  is a cluster point of  $A$ .

Contd.

- (d) Write sequential criterion for continuity.
- (e) What do you mean by an unbounded function on a set ?
- (f) Let  $A, B \subseteq \mathbb{R}$  and let  $f: A \rightarrow \mathbb{R}$  be continuous on  $A$  and let  $g: B \rightarrow \mathbb{R}$  be continuous on  $B$ . Under what condition  $g \circ f: A \rightarrow \mathbb{R}$  is continuous on  $A$  ?
- (g) "If a function is continuous then it is uniformly continuous."  
(Write True or False)
- (h) If functions  $f_1, f_2, \dots, f_n$  are differentiable at  $c$ , write the expression for  $(f_1, f_2, \dots, f_n)'(c)$ .
- (i) The function  $f(x) = x$  is defined on the interval  $I = [0, 1]$ . Is 0 a relative maximum of  $f$  ?
- (j) Define Taylor's polynomial for a function  $f$  at a point  $x_0$ , supposing  $f$  has an  $n$ th derivative at  $x_0$ .

2. Answer the following questions :  $2 \times 5 = 10$

- (a) Use  $\varepsilon - \delta$  definition of limit to show that  $\lim_{x \rightarrow 0} \frac{x^2}{|x|} = 0$ .
- (b) Show that the absolute value function  $f(x) = |x|$  is continuous at every point  $c \in \mathbb{R}$ .
- (c) Give an example of a function  $f: [0, 1] \rightarrow \mathbb{R}$  that is discontinuous at every point of  $[0, 1]$ , but  $|f|$  is continuous on  $[0, 1]$ .
- (d) "Continuity at a point is not a sufficient condition for the derivative to exist at that point." Justify your answer.
- (e) Show that  $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sqrt{x}} = 0$ .

3. Answer any four parts :  $5 \times 4 = 20$

- (a) Prove that a number  $c \in \mathbb{R}$  is a cluster point of a subset  $A$  of  $\mathbb{R}$  if and only if there exists a sequence  $\{a_n\}$  in  $A$  such that  $\lim a_n = c$  and  $a_n \neq c$  for all  $n \in \mathbb{N}$ .

(b) Show that (using  $\varepsilon - \delta$  definition of limit)

$$\lim_{x \rightarrow 2} \frac{x^3 - 4}{x^2 + 1} = \frac{4}{5}$$

(c) Prove that if  $I = [a, b]$  is a closed bounded interval and if  $f: I \rightarrow \mathbb{R}$  is continuous on  $I$  then  $f$  is bounded on  $I$ .

(d) Show that if  $f$  and  $g$  are uniformly continuous on a subset  $A$  of  $\mathbb{R}$  then  $f + g$  is uniformly continuous on  $A$ .

(e) Suppose that  $f$  is continuous on a closed interval  $I = [a, b]$  and that  $f$  has a derivative in the open interval  $(a, b)$ . Then there exists at least one point  $c$  in  $(a, b)$  such that

$$f(b) - f(a) = f'(c)(b - a).$$

(f) Let  $f: I \rightarrow \mathbb{R}$  be differentiable on the interval  $I$ . Then prove that  $f$  is increasing if and only if  $f'(x) \geq 0$  for all  $x \in I$ .

4. Answer **any four** parts :  $10 \times 4 = 40$

(a) Prove that a real valued function  $f$  is continuous at  $c \in \mathbb{R}$  if and only if whenever every sequence  $\{c_n\}$ , converging to  $c$ , then corresponding sequence  $\{f(c_n)\}$  converges to  $f(c)$ .

(b) (i) Show that every infinite bounded subset of  $\mathbb{R}$  has at least one limit point.

(ii) Let  $A \subseteq \mathbb{R}$ , let  $f: A \rightarrow \mathbb{R}$  and let  $c \in \mathbb{R}$  be a cluster point of  $A$ . If  $a \leq f(x) \leq b$  for all  $x \in A$ ,  $x \neq c$  and if  $\lim_{x \rightarrow c} f(x)$  exist then prove that

$$a \leq \lim_{x \rightarrow c} f \leq b.$$

(c) (i) Let  $I = [a, b]$  be a closed bounded interval. Let  $f: I \rightarrow \mathbb{R}$  be such that  $f$  is continuous. Prove that  $f$  is uniformly continuous on  $[a, b]$ .

- (ii) Show that the function  $f(x) = \frac{1}{x^2}$  is uniformly continuous on  $A = [1, \infty)$ . 5
- (d) Let  $I = [a, b]$  be a closed bounded interval and let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . Then  $f$  has an absolute maximum and an absolute minimum on  $I$ .
- (e) (i) Let  $I$  be a closed bounded interval and let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . Then the set  $f(I) = \{f(x) : x \in I\}$  is a closed bounded interval. 6
- (ii) Let  $A, B \subseteq \mathbb{R}$  and let  $f : A \rightarrow \mathbb{R}$  and  $g : B \rightarrow \mathbb{R}$  be functions such that  $f(A) \subseteq B$ . If  $f$  is continuous at a point  $c \in A$  and  $g$  is continuous at  $b = f(c) \in B$ , then show that the composition  $g \circ f : A \rightarrow \mathbb{R}$  is continuous at  $c$ . 4

- (f) (i) Let  $I = [a, b]$  and let  $f : I \rightarrow \mathbb{R}$  be continuous on  $I$ . If  $f(a) < 0 < f(b)$  or if  $f(a) > 0 > f(b)$ , then prove that there exists a number  $c \in (a, b)$  such that  $f(c) = 0$ . 6
- (ii) Use the definition to find the derivative of the function  $f(x) = \frac{1}{\sqrt{x}}$  for  $x > 0$ . 4
- (g) (i) State and prove Taylor's theorem.  $2+5=7$
- (ii) Using the Mean Value theorem prove that  $|\sin x - \sin y| \leq |x - y|$  for all  $x, y$  in  $\mathbb{R}$ . 3
- (h) (i) Show that  $1 - \frac{1}{2}x^2 \leq \cos x$  for all  $x \in \mathbb{R}$  5
- (ii) Evaluate  $\lim_{x \rightarrow 0} \left( \frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$  5